$SO(5)$ structure of p-wave superconductivity for the spin-dipole interaction model

2002 J. Phys. A: Math. Gen. 35 L7

(http://iopscience.iop.org/0305-4470/35/1/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L7–L11 PII: S0305-4470(02)26061-8

LETTER TO THE EDITOR

$SO(5)$ structure of p-wave superconductivity for the spin–dipole interaction model

Hong-Biao Zhang[1,2], **Mo-Lin Ge**[2] **and Kang-Xue**[1]

[1] Department of Physics, Northeast Normal University, Changchun, Jilin 130024, People's Republic of China

[2] Theoretical Physics Division, Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

E-mail: hbzhang@eyou.com

Received 13 June 2001, in final form 9 November 2001
Published 14 December 2001
Online at stacks.iop.org/JPhysA/35/L7

Abstract

A closed $SO(5)$ algebraic structure in the pure p-wave superconductivity is found. It can help to diagonalize the the mean-field form of the Hamiltonian by making use of the Bogoliubov rotation instead of the Balian–Werthamer approach. We point out that the eigenstate is nothing but an $SO(5)$-coherent state with fermionic realization. By applying the approach to the Hamiltonian with Leggett dipole interaction the consistency between the diagonalization and gap equation is proved through the double-time Green function. The relationship between the s- and p-wave superconductivities turns out to be realized through Yangian algebra, a new type of infinite-dimensional algebra.

PACS numbers: 03.65.-w, 03.65.Fd, 74.20.Fg, 74.20.-z

The p-wave superconductivitiy theories and their applications to liquid $^3\mathrm{H}_e$ have intensively been studied in many early publications, for example, in [1–5]. As was pointed out by Anderson and Brinkman [1] the Balian–Werthamer (BW) formalism [6] underlies all the following models in the field. The Hamiltonian takes the Anderson reduced form $H = H_0 + V$

$$H_0 = \sum_{\boldsymbol{k},\alpha} \epsilon_{\boldsymbol{k}} n_{\boldsymbol{k}\alpha} \qquad V = \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k}'\alpha,\beta} V_{\boldsymbol{k}\boldsymbol{k}'} a^+_{\boldsymbol{k}'\alpha} a^+_{-\boldsymbol{k}'\beta} a_{-\boldsymbol{k}\beta} a_{\boldsymbol{k}\alpha} \tag{1}$$

where $\epsilon_{\boldsymbol{k}} = \frac{k^2}{2m} - \mu$, $\alpha, \beta = \uparrow, \downarrow$, and for the p-wave $V_{\boldsymbol{k}\boldsymbol{k}'} = -3V_1(k, k')\boldsymbol{n} \cdot \boldsymbol{n}'$ $\left(\boldsymbol{n} = \frac{\boldsymbol{k}}{k}\right)$. To explain the p-pair interaction Leggett [2] introduced a useful algebra, which is not closed. Meanwhile, a dipole type of interaction was proposed that naturally distinguishes the energy difference for the ABM and BW phases [2–4] and gives the correct spin dynamics. All the theories appear to work perfectly, but with growing interest in the applications of the current algebraic method it is still desirable to gain greater understanding. In this paper we would like to show the following results. (a) The sets obeying $S_i(\boldsymbol{k}) = a^+_{\boldsymbol{k}\alpha}(\sigma_i)_{\alpha\beta} a_{\boldsymbol{k}\beta}$

and $T_i(\boldsymbol{k}) = a_{-k\alpha}(\sigma_2\sigma_i)_{\alpha\beta}a_{k\beta}$ as well as their conjugates [2] form an $SO(5)$ algebra ($i = 0, 1, 2, 3, \sigma_0 = \mathbf{1}$ and summation over the repeat α and β) that is larger than the usual $U(1) \otimes SU^{(L)} \otimes SU^{(S)}$ as shown in [3]. Equipped with the algebraic structure for the lower pair excitation we then diagonalize the Hamiltonian equation (1) that together with the invariance $U(1) \otimes SU^{(L)} \otimes SU^{(S)}$ by using the algebraic average method (AAM, general Bogoliubov rotation) yield all the known results. (b) We show that the eigenfunction of equation (1) is an $SO(5)$-coherent state with fermionic realization, hence the coherence property related to equation (1) originates in the closed $SO(5)$ structure. (c) The above calculation can be applied to the dipole interaction Hamiltonian of Leggett. There appears a nontrivial consistency between the diagonalization of the microscopic form of H_D by AAM and the gap equation. (d) Finally, in contrast to the $SO(5)$ unification of Zhang *et al* [8, 9] we attempt to find what price we have to pay in order to form an assumed unification involving both the s- and p-wave superconductivities (as shown in [2]), instead of the claimed transition between s-wave superconductivity and the AF phase shown in [8, 9].

(1) Observing the algebra defined in [2], if $T_0(\boldsymbol{k})$ is picked up, it becomes closed. Defining $\bar{\boldsymbol{S}}(\boldsymbol{k}) = \frac{1}{2}[\boldsymbol{S}(\boldsymbol{k}) + \boldsymbol{S}(-\boldsymbol{k})]$, $Q(\boldsymbol{k}) = \frac{1}{2}(S_0(\boldsymbol{k}) + S_0(-\boldsymbol{k}) - 2)$ it can be checked that the set $(Q(\boldsymbol{k}), \bar{\boldsymbol{S}}(\boldsymbol{k}), \boldsymbol{T}(\boldsymbol{k}), \boldsymbol{T}^\dagger(\boldsymbol{k}))$ forms an $SO(5)$ algebra

$$[I_{ab}(\boldsymbol{k}), I_{cd}(\boldsymbol{k}')] = -\mathrm{i}\delta(\boldsymbol{k} - \boldsymbol{k}')(\delta_{ac}I_{bd}(\boldsymbol{k}) + \delta_{bd}I_{ac}(\boldsymbol{k}) - \delta_{ad}I_{bc}(\boldsymbol{k}) - \delta_{bc}I_{ad}(\boldsymbol{k}))$$

where $I_{ab}(\boldsymbol{k}) = -I_{ba}(\boldsymbol{k})(a, b = 1, 2, 3, 4, 5)$ takes the form

$$\begin{pmatrix} 0 & & & & \\ -\frac{1}{2}(T_x^\dagger(\boldsymbol{k}) + T_x(\boldsymbol{k})) & 0 & & & \\ -\frac{1}{2}(T_y^\dagger(\boldsymbol{k}) + T_y(\boldsymbol{k})) & -\bar{S}_z(\boldsymbol{k}) & 0 & & \\ -\frac{1}{2}(T_z^\dagger(\boldsymbol{k}) + T_z(\boldsymbol{k})) & \bar{S}_y(\boldsymbol{k}) & -\bar{S}_x(\boldsymbol{k}) & 0 & \\ Q(\boldsymbol{k}) & \frac{1}{2\mathrm{i}}(T_x(\boldsymbol{k}) - T_x^\dagger(\boldsymbol{k})) & \frac{1}{2\mathrm{i}}(T_y(\boldsymbol{k}) - T_y^\dagger(\boldsymbol{k})) & \frac{1}{2\mathrm{i}}(T_z(\boldsymbol{k}) - T_z^\dagger(\boldsymbol{k})) & 0 \end{pmatrix}. \tag{2}$$

The Hamiltonian equation (1) can then be written in the form

$$H = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}}(Q(\boldsymbol{k}) + 1) + \frac{1}{4}\sum_{\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{k}'\boldsymbol{k}}\boldsymbol{T}^\dagger(\boldsymbol{k}) \cdot \boldsymbol{T}(\boldsymbol{k}'). \tag{3}$$

Note that the mean-field approximation is enough to obtain the gap equation since we work in the equilibrium state. Using [10]

$$AB \simeq A\langle B\rangle + \langle A\rangle B - \langle A\rangle\langle B\rangle$$

equation (3) can be linearized with respect to $SO(5)$ generators

$$H_{mf} = \sum_{\boldsymbol{k}}\{H(\boldsymbol{k}) - E_*(\boldsymbol{k})\}$$

with

$$H(\boldsymbol{k}) = \epsilon_{\boldsymbol{k}}Q(\boldsymbol{k}) + \boldsymbol{\Delta}(\boldsymbol{k}) \cdot \boldsymbol{T}^\dagger(\boldsymbol{k}) + \boldsymbol{\Delta}^*(\boldsymbol{k}) \cdot \boldsymbol{T}(\boldsymbol{k}) \tag{4}$$

$$E_*(\boldsymbol{k}) = \epsilon_{\boldsymbol{k}} - \boldsymbol{\Delta}(\boldsymbol{k}) \cdot \langle \boldsymbol{T}^\dagger(\boldsymbol{k})\rangle \tag{5}$$

where $\boldsymbol{\Delta}(\boldsymbol{k}) = \frac{1}{4}\sum_{\boldsymbol{k}'} V_{\boldsymbol{k}'\boldsymbol{k}}\langle \boldsymbol{T}(\boldsymbol{k})\rangle$ and $\langle \ldots \rangle$ represents the average over both quantum states and thermodynamics. We emphasized that the set $\Lambda = \left\{\frac{\mathrm{i}}{\sqrt{2}}T_3(\boldsymbol{k}), \frac{-\mathrm{i}}{\sqrt{2}}T_3^\dagger(\boldsymbol{k}), -Q(\boldsymbol{k})\right\}$ i.e. $\left\{-\mathrm{i}\sqrt{2}\pi_z, \mathrm{i}\sqrt{2}\pi_z^\dagger, -Q\right\}$ in [8, 9] forms the quasi-spin Λ for pairs. Λ does not commute with spin operators $\boldsymbol{S}(\boldsymbol{k})$ that give rise to $T_\pm^\dagger(\boldsymbol{k})$ and $T_\pm(\boldsymbol{k})$ which are beyond two $SU(2)$ and the total set forms $SO(5)$. In order to perform the diagonalization of equation (4) we introduce the unitary transformation such that $W^\dagger(\xi_{\boldsymbol{k}})H(\boldsymbol{k})W(\xi_{\boldsymbol{k}})$ becomes diagonal for any $\boldsymbol{k}$. Following the general strategy [7] we introduce the $SO(5)$-coherent operator:

$$W(\xi_{\boldsymbol{k}}) = \exp\{\xi_{\boldsymbol{k}}[\boldsymbol{d}(\boldsymbol{n}) \cdot \boldsymbol{T}^\dagger(\boldsymbol{k})] - \text{h.c.}\} \tag{6}$$

where $\xi_k = r_k e^{i\lambda_k}$, $\boldsymbol{d}(\boldsymbol{n}) = (\sin\Theta_k \cos\Phi_k, \sin\Theta_k \sin\Phi_k, \cos\Theta_k)$, Θ_k and Φ_k are angulars in spin space for a given momentum $\boldsymbol{k}$. λ_k is a parameter to be determined using the gap equation. Taking the commutation relations for $SO(5)$ into account after lengthy but elementary calculations we derive

$$W(\xi_k)^{-1} H(\boldsymbol{k}) W(\xi_k) = -E_k Q(\boldsymbol{k}) \qquad E_k = \sqrt{\epsilon_k^2 + |\boldsymbol{\Delta}(\boldsymbol{k})|^2} \tag{7}$$

where

$$\tan 2r_k = \frac{|\boldsymbol{\Delta}(\boldsymbol{k})|}{\epsilon_k} \qquad \boldsymbol{\Delta}(\boldsymbol{k}) = \frac{1}{4}\sum_{\boldsymbol{k}'} V_{k'k}\langle \boldsymbol{T}(\boldsymbol{k})\rangle = -\frac{1}{2}|\boldsymbol{\Delta}(\boldsymbol{k})|e^{i\lambda_k}\boldsymbol{d}(\boldsymbol{n}). \tag{8}$$

The eigenstate is given by

$$|\xi\rangle = \otimes_k |\xi_k\rangle \qquad |\xi_k\rangle = W(\xi_k)|\mathrm{vac}\rangle. \tag{9}$$

At temperature $T = 0$, the vacuum state is $|\mathrm{vac}\rangle = |0,0\rangle_k \equiv |n_{k\alpha} = 0, n_{-k\alpha} = 0\rangle$. The expectation value $\langle \boldsymbol{T}(\boldsymbol{k})\rangle = \langle \xi_k|\boldsymbol{T}(\boldsymbol{k})|\xi_k\rangle = \sin 2r_k e^{i\lambda_k}\boldsymbol{d}(\boldsymbol{n})$ yields the well known gap equation at $T = 0$:

$$\boldsymbol{\Delta}(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_{k'k}\frac{\boldsymbol{\Delta}(\boldsymbol{k}')}{2E_{k'}}. \tag{10}$$

To satisfy equation (10) we simply choose $\lambda_k = \lambda =$ constant henceforth, for finite temperature, making use of the double-time Green function we obtain $\langle n_{k\alpha}\rangle = \frac{1}{2}[1 - \frac{\epsilon_k}{E_k}\tanh(\frac{1}{2}\beta E_k)]$ and $\langle \boldsymbol{T}(\boldsymbol{k})\rangle = \langle \xi_k|\boldsymbol{T}(\boldsymbol{k})|\xi_k\rangle = \sin 2r_k e^{i\lambda}\tanh(\frac{1}{2}\beta E_k)\boldsymbol{d}(\boldsymbol{n})$ where $\beta = \frac{1}{kT}$ and k is the Boltzmann constant. Therefore the gap equation reads

$$\boldsymbol{\Delta}(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_{kk'}\frac{\boldsymbol{\Delta}(\boldsymbol{k}')}{2E_{k'}}\tanh\left(\frac{1}{2}\beta E_{k'}\right). \tag{11}$$

The $SO(5)$ coherent state $|\xi_k\rangle$ equation (9) gives

$$\begin{aligned}|\xi_k\rangle = W(\xi_k)|0,0\rangle &= \cos^2 r_k|0,0\rangle - e^{i2\lambda}\sin^2 r_k|\uparrow\downarrow,\uparrow\downarrow\rangle \\ &+ \frac{i}{2}e^{i\lambda}\sin 2r_k\{\cos\Theta_k(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle) \\ &- \sin\Theta_k e^{-i\Phi_k}|\uparrow,\uparrow\rangle + \sin\Theta_k e^{i\Phi_k}|\downarrow,\downarrow\rangle\}.\end{aligned} \tag{12}$$

Let us distinguish two cases: in the BW phase, to satisfy the gap equation it holds $\boldsymbol{d}(\boldsymbol{n}) = \boldsymbol{n}$, $\Theta_k = \theta_k$ and $\Phi_k = \psi_k$ that correspond intuitively to a Cooper pair with total angular momentum $J = 0$ and has an isotropic gap, $|\boldsymbol{\Delta}(\boldsymbol{k})|e^{i\lambda} = c$-number. The wavefunction of the BW solution in conventional notation reads

$$\begin{aligned}|\xi_k\rangle = \frac{E_k + \epsilon_k}{2E_k}|0,0\rangle &- e^{i2\lambda}\frac{E_k - \epsilon_k}{\boldsymbol{2E_k}}|\uparrow\downarrow,\uparrow\downarrow\rangle \\ &- e^{i\lambda}\frac{i|\boldsymbol{\Delta}(\boldsymbol{k})|}{2E_k}\sqrt{\frac{8\pi}{3}}\{Y_{11}|\downarrow,\downarrow\rangle - \frac{1}{\sqrt{2}}Y_{10}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle) + Y_{1-1}|\uparrow,\uparrow\rangle\}.\end{aligned} \tag{13}$$

In the AM case, there is another solution of the gap equation obtained by taking $|\boldsymbol{\Delta}(\boldsymbol{k})|e^{i(\lambda+\frac{\pi}{2})} = \boldsymbol{Y}_{11}$ and $\sin\Theta_k = 0$, it is the non-ESP state

$$|\xi_k\rangle = \frac{E_k + \epsilon_k}{2E_k}|0,0\rangle + e^{i2\lambda}\frac{E_k - \epsilon_k}{2E_k}|\uparrow\downarrow,\uparrow\downarrow\rangle + \frac{Y_{11}}{2E_k}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle). \tag{14}$$

However, the solution for $\cos\Theta_k = 0$ appears only under an applied magnetic field. For instance, when $\boldsymbol{B} = \mu B\boldsymbol{e}_z$, the Hamiltonian becomes

$$H_B = \sum_{\boldsymbol{k}} H(\boldsymbol{k}) - \mu B\sum_{\boldsymbol{k}}\left(a^+_{k\uparrow}a_{k\uparrow} - a^+_{k\downarrow}a_{k\downarrow}\right)$$

and

$$W^\dagger H_B W = \frac{1}{2}E_k\left(1+\frac{\mu B}{\epsilon_k}\right)(n_{k\downarrow}+n_{-k\downarrow}) + \frac{1}{2}E_k\left(1-\frac{\mu B}{\epsilon_k}\right)(n_{k\uparrow}+n_{-k\uparrow}) - E_k.$$

Through the double-time Green function we may calculate that the non-vanishing components of $\boldsymbol{\Delta}(\boldsymbol{k})$ are

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) = \frac{1}{4}\sum_{k'} V_{kk'}\frac{|\boldsymbol{\Delta}(\boldsymbol{k}')|}{2E_{k'}}\mathrm{e}^{\mathrm{i}\lambda}\mathrm{e}^{\mathrm{i}\Phi_{k'}}\tanh\left[\frac{1}{2}\beta E_{k'}\left(1-\frac{\mu B}{\epsilon_{k'}}\right)\right]$$

and

$$\Delta_{\downarrow\downarrow}(\boldsymbol{k}) = \frac{1}{4}\sum_{k'} V_{kk'}\frac{|\Delta(\boldsymbol{k}')|}{2E_{k'}}\mathrm{e}^{i\lambda}\mathrm{e}^{-i\Phi_{k'}}\tanh\left[\frac{1}{2}\beta E_{k'}\left(1+\frac{\mu B}{\epsilon_{k'}}\right)\right].$$

The eigenstate is $\sim Y_{11}\,(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$. If we study s-superconductivity by AAM, then coherence comes from the $SU(2)$ coherent operators in [7]. The AAM is exactly the usual Bogoliubov transformation.

(2) The dipole interaction was proposed to describe the spin dynamics for liquid $^3\mathrm{H}_e$. The computation in [2] is based on $\langle T(\boldsymbol{k})\rangle = \sum_{i=1}^{3} T_i^\alpha n_i$. However, in the microscopic-form of the dipole interaction Hamiltonian

$$H_D = \frac{2\pi\gamma^2}{3}\sum_{kk} V_{kk'}(\boldsymbol{T}^\dagger(\boldsymbol{k})\cdot\boldsymbol{T}(\boldsymbol{k}') - 3\hat{\boldsymbol{q}}\cdot\boldsymbol{T}^\dagger(\boldsymbol{k})\hat{\boldsymbol{q}}\cdot\boldsymbol{T}(\boldsymbol{k}')) \tag{15}$$

where $\hat{\boldsymbol{q}}$ is a unit vector along $\boldsymbol{n}-\boldsymbol{n}'$. The average formula shown in [2] can no longer be used, since the operator cannot be expended by n_i. However, the $SO(5)$ AAM procedure works for equation (15) with the redefinition $\Delta_i = \sum_{j=1}^{3}(\delta_{ij} - 3\hat{q}_i\hat{q}_j)\langle T_j\rangle$. Repeating the process in (1) we find that $\boldsymbol{\Delta}(\boldsymbol{k})$ satisfies the same gap equation (11) and the eigenstates take the same form for given $\boldsymbol{\Delta}(\boldsymbol{k})$ satisfying equation (11), i.e. equation (15) can be diagonalized with the following relation instead of equation (8):

$$\begin{pmatrix}\Delta_{\uparrow\uparrow}(\boldsymbol{k})\\ \sqrt{2}\Delta_{\uparrow\downarrow}(\boldsymbol{k})\\ \Delta_{\downarrow\downarrow}(\boldsymbol{k})\end{pmatrix} = \frac{4\pi\gamma^2}{3}\sum_{k'} V_{k'k}\left\{\tfrac{1}{2}I + \tfrac{3}{2}D^{j=1}(\alpha=\psi_{kk'},\beta=2\omega_{kk'},\gamma=\pi-\psi_{kk'})\right\} \times\begin{pmatrix}\left\langle\frac{1}{2\mathrm{i}}T_-(\boldsymbol{k}')\right\rangle\\ \left\langle\frac{\mathrm{i}}{\sqrt{2}}T_z(\boldsymbol{k}')\right\rangle\\ \left\langle\frac{\mathrm{i}}{2}T_+(\boldsymbol{k}')\right\rangle\end{pmatrix} \tag{16}$$

where the $D^{j=1}(\alpha,\beta,\gamma)$ is the Wigner rotation function with the Euler angles α, β and γ and $\hat{\boldsymbol{q}} = q(\sin\omega_{kk'}\cos\psi_{kk'}, \sin\omega_{kk'}\sin\psi_{kk'}, \cos\omega_{kk'})$. The relations for $\omega_{kk'}$, $\psi_{kk'}$ and α, β, γ have been indicated in equation (16). The H_D works well in spin dynamics, but the consistency condition for the diagonalization of H_D and the gap equation, to our knowledge, have not been proved before. Now equation (8) is replaced by (16) for the Hamiltonian equation (15). This means that all the discussion in (1) can be transplanted literally for $\boldsymbol{\Delta}(\boldsymbol{k})$.

(3) It seems that $T_\pm$ (or $\sim\pi_\pm$ in [8, 9]) in $SO(5)$ may give rise to the transition between superconductivity and the AF state based on the argument given in [8, 9]. However, it is not the case in the present model. This is not only because there is no $SO(5)$ invariance for H or H_D, but also for deeper reasons. Observing the gap equation for $V_{kk'} \sim P_0$ (constant) and $V_{kk'} \sim P_1 \sim \boldsymbol{n}\cdot\boldsymbol{n}'$, the corresponding wavefunction $\Psi_0 \sim Y_{00}\chi_{00}$ and $\Psi_1 \sim$ equation (13) where χ_{00} is spin singlet. In our case, the generators of $SO(5)$ work only within p-superconductivity, i.e. not with s-superconductivity. If we assume there is a

transition between Ψ_0 and Ψ_1, i.e. the form of the gap equation is preserved, but with different potentials $\sim P_0$ and P_1, their corresponding wavefunctions are written as follows:

$$\Psi_0 \sim \frac{Y_{00}}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Psi_1 \sim \frac{1}{\sqrt{3}}\begin{pmatrix} Y_{1-1} & \frac{-1}{\sqrt{2}}Y_{10} \\ \frac{-1}{\sqrt{2}}Y_{10} & Y_{11} \end{pmatrix} = \frac{1}{\sqrt{8\pi}}\begin{pmatrix} \hat{k}_- & -\hat{k}_z \\ -\hat{k}_z & -\hat{k}_+ \end{pmatrix}. \tag{17}$$

The connection should be beyond Lie algebra. We find that such a 'transition' may be performed through Yangian theory [11–13]. Actually,

$$\begin{aligned} \hat{k}_\pm J_\mp \Psi_0 &= \left(\mu_2 - \mu_1 + \frac{h}{2}\right) Y_{1\pm 1}\chi_{1\mp 1} \\ \hat{k}_z J_z \Psi_0 &= -\frac{1}{2}\left(\mu_2 - \mu_1 + \frac{h}{2}\right) Y_{10}\chi_{10} \end{aligned} \tag{18}$$

where $J_\alpha = \mu_1 S_\alpha \otimes 1 + \mu_2 1 \otimes S_\alpha - \frac{\mathrm{i}h}{4}\epsilon_{\alpha\beta\gamma}(S_\beta \otimes S_\gamma - S_\gamma \otimes S_\beta)$ $(\alpha, \beta, \gamma = 1, 2, 3)$ and S_α are the spin operators. μ_1 and μ_2 are arbitrary constants allowed by Yangian representation theory and play the crucial role in the Yangian [14] representation. $[S_\alpha, J_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma} J_\gamma$ and J_γ obey the nonlinear commutation relations [11–15]. The set $\{S_\alpha, J_\beta\}$ forms the Yangian associated with $SU(2)$ denoted by $Y(SU(2))$. Note that J_α act on the quantum tensor space only. If the set $\{\boldsymbol{S}, \boldsymbol{J}\}$ satisfies the Yangian, so does $\boldsymbol{J} + \eta \boldsymbol{S}$, where η is an arbitrary constant that is called the translation of the Yangian. By taking an appropriate translation constant we have

$$(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})\Psi_0 = \frac{\sqrt{3}}{2}\left(\mu_2 - \mu_1 + \frac{h}{2}\right)\Psi_1 \qquad (\hat{\boldsymbol{k}} \cdot \boldsymbol{J})\Psi_1 = 0. \tag{19}$$

Yangian algebra is an infinite algebra. Therefore any attempt to unify the superconductivity with different l-waves should be through infinite algebra. For a simply physical realization of $Y(SU(2))$, see [15].

(4) In conclusion we believe that the AAM provides a useful approach to discuss physics concerning pair-particles, especially for the nature of coherence and consistency between the diagonalization of the given Hamiltonian and gap equation through the double-time Green function. Further, this algebraic method may be extended to Yangian algebra that is natural to describe the transition between different condensates.

This work is, in part, supported by the National Natural Science Foundation of China.

References

[1] Anderson P W and Brinkman W F 1978 *Theory of Anisotropic Superfluidity in* ^3He Part II, ed K H Bennemann and J B Ketterson (New York: Wiley) ch 3

[2] Leggett A J 1974 *Ann. Phys., NY* **85** 11–55
Leggett A J 1975 *Rev. Mod. Phys.* **47** 331–413

[3] Volovik G E 1992 *Exotic Properties of Superfluid* ^3He (Singapore: World Scientific)

[4] Lee D M 1997 *Rev. Mod. Phys.* **69** 645–64

[5] Balian R and Werthamer N R 1963 *Phys. Rev.* **131** 1553–64

[6] Anderson P W 1958 *Phys. Rev.* **112** 1900

[7] Zhang W, Feng D and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867

[8] Zhang S C 1997 *Science* **275** 1089

[9] Scalapinp D J, Zhang S C and Hanke W 1998 *Phys. Rev.* B **58** 443

[10] Solomon A I, Feng Y and Penna V 1999 *Phys. Rev.* B **60** 3044

[11] Drinfeld V 1985 *Sov. Math.–Dokl.* **32** 254

[12] Drinfeld V 1985 *Sov. Math.–Dokl.* **36** 212

[13] Drinfeld V 1986 *Quantum Group* (Berkely: PICM) pp 269–91

[14] Chari V and Pressley A 1990 Yangian and R-matrix *L'Enseignement Matne-Matique* **36** 267–302
Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[15] Ge M L, Xue K and Cho Y M 1998 *Phys. Lett.* A **249** 358